Journal of Engineering Physics and Thermophysics, Vol. 80, No. 1, 2007

STABILITY OF THE AIR CONVECTION IN A TWO-LAYER COVER OF SNOW. I. SYSTEM OF LINEARIZED EQUATIONS FOR THERMAL AIR CONVECTION

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The problem on determination of the critical Rayleigh number at which the air contained in the pores of the snow in a snow cover consisting of two layers with different thermophysical and structural parameters becomes unstable was formulated.

It was established in [1] that, in the case where the temperature gradient in a homogeneous cover of snow exceeds any critical value, the humid air contained in the snow pores begins to execute a convective motion. This motion substantially influences the processes of heat and mass transfer in the snow cover and plays a significant role in the metamorphism of the snow and in the formation of its structure.

A snow cover formed as a result of only one snowfall with a wind can be layered, and a snow cover formed as a result of several snowfalls has, deliberately, a layered structure. Each newly fallen snow is transformed in the process of metamorphism and forms an indivisible, fairly homogeneous and lengthy structure with definite thermophysical and strength properties. As a result of the formation of such a structure, the temperature field of the snow cover changes and vapor flows appear at the boundaries of the snow layers.

Thus, in the majority of cases, intensive moisture flows appear and snow structures are formed in a snow cover consisting of two or more layers of snow with more or less homogeneous thermophysical parameters. The study of the conditions under which a convective air motion arises in multilayer snow covers is of both scientific and practical interest.

In the present work, we derived a system of linearized equations defining the thermal convection of air in a two-layer snow and formulated the problem on determination of the critical parameters at which the air contained in the snow pores becomes unstable.

Let us consider a snow cover consisting of two layers, in which the lower layer is bounded by the horizontal planes $z = 0$ and $z = h_1$ and the upper layer is bounded by the planes $z = h_1$ and $z = H$. The parameters of the lower snow layer will be denoted by the index "1", and the parameters of the upper snow layer will be denoted by index "2." The z axis represents a vertical line directed upward from the lower base of the snow cover (see Fig. 1).

The linearized equation defining the air motion and the equation for the heat conduction in each snow layer have the form [1]

$$\frac{\partial \mathbf{V}_i}{\partial t} = -\frac{1}{\rho_0}\nabla p_i - \frac{\nu}{\sigma_i}\mathbf{V}_i + \beta f_i g \theta_i \mathbf{e}_z\,, \qquad (1)$$

$$\frac{\partial \theta_i}{\partial t} + M_i \mathbf{V}_i \nabla \theta_{i\text{st}} = \chi_i \Delta \theta_i\,, \qquad (2)$$

where

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Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 80, No. 1, pp. 107–112, January–February, 2007. Original article submitted June 2, 2005.

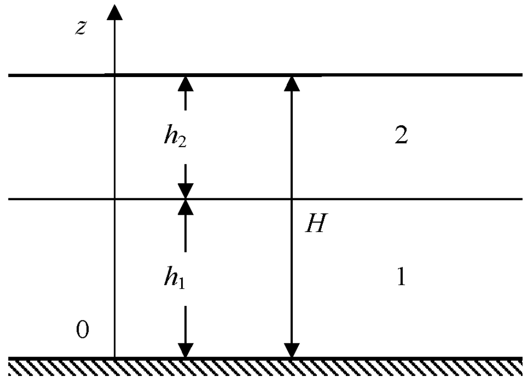

Fig. 1. Scheme of a two-layer snow cover.

$$M_i = \frac{\rho_0 f_i c_p}{\rho_{si} c_{si}} \left(1 + \frac{L_s^2 \rho_{w0}}{\rho_0 R_w c_p T_0} \right), \quad i = 1, 2\,;$$

here, the quantities M_1 and M_2 define the heat transferred by a flow of moisture-saturated air contained in the snow pores; p_i and θ_i are small perturbations of the snow pressure and temperature, and $\theta_{ist}(z)$ are stationary temperature perturbations in the lower and upper layers of the snow cover at the boundary conditions $\Delta\theta_{1st}(0) = \theta_{10}$ and $\Delta\theta_{2st}(0) = \theta_{20}$ and the conjugation conditions $\theta_{1st}(h_1) = \theta_{2st}(h_1)$ and $\lambda_1 \left.\frac{d\theta_{1st}}{dz}\right|_{z=h_1} = \lambda_1 \left.\frac{d\theta_{1st}}{dz}\right|_{z=h_1}$:

$$\theta_{1st} = -\frac{\theta_{10} - \theta_{20}}{h_1 + \frac{\lambda_1}{\lambda_2} h_2} z + \theta_{10}\,, \tag{3}$$

$$\theta_{2st} = -\frac{\theta_{10} - \theta_{20}}{h_2 + \frac{\lambda_2}{\lambda_1} h_1} (H - z) + \theta_{20}\,. \tag{4}$$

The continuity equation for the air contained in the snow pores has the form

$$\operatorname{div} \mathbf{V}_1 = 0\,, \tag{5}$$

$$\operatorname{div} \mathbf{V}_2 = 0\,. \tag{6}$$

The linear system of equations (1)–(6) defines the thermal convection of air in a two-layer snow cover. The boundary and conjugation conditions for this system are formulated in the following way. The lower boundary (ground) of the snow cover is impenetrable; therefore, the vertical air-flow velocity component at this boundary is equal to zero:

$$\left. V_{1z} \right|_{z=0} = 0\,. \tag{7}$$

When the upper boundary of the snow cover is penetrable for air, it is assumed that the horizontal pressure gradient is equal to zero at $z = H$. Since V_x and V_y are equal to zero at the upper surface of the snow cover, from the continuity equation we obtain that

$$\left.\frac{\partial V_{2z}}{\partial z}\right|_{z=H} = 0\,. \tag{8}$$

If the upper surface of the snow cover is impenetrable for air, which can take place in the case where it is covered with a thin ice layer,

$$V_{2z}\,(H) = 0\,. \tag{9}$$

The temperatures of the lower and upper boundaries are constant and, therefore, temperature perturbations are absent:

$$\theta_1\,(0) = 0\,, \quad \theta_2\,(H) = 0\,. \tag{10}$$

Moreover, the continuity conditions should be fulfilled for the temperatures, mass flows, and heat flows at the interfaces between the layers:

$$\theta_1\,(h_1) = \theta_2\,(h_1)\,, \tag{11}$$

$$f_1 V_{1z}\sigma_1\Big|_{z=h_1} = f_2 V_{2z}\sigma_2\Big|_{z=h_1}\,, \tag{12}$$

$$\left.\left(\lambda_1\frac{d\theta_1}{dz} + \rho_0 c_p f_1 \theta_{1\mathrm{st}} V_{1z}\right)\right|_{z=h_1} = \left.\left(\lambda_2\frac{d\theta_2}{dz} + \rho_0 c_p f_2 \theta_{2\mathrm{st}} V_{2z}\right)\right|_{z=h_1}\,. \tag{13}$$

Near the instability boundary, where the velocities of the convective air flows are small ($V_z \sim 10^{-7}$ m/sec and smaller), the Peclet number Pe = $MV_2H/\chi_\mathrm{s} << 1$ and, therefore, the terms including velocities in the continuity equation (13) can be disregarded:

$$\lambda_1\left.\frac{d\theta_1}{dz}\right|_{z=h_1} = \lambda_2\left.\frac{d\theta_2}{dz}\right|_{z=h_1}\,. \tag{14}$$

In the case where condition (13) is changed to condition (14), the problem becomes much simpler and there appears a possibility of solving it analytically. To write Eqs. (1) and (2) in the dimensionless form, we will introduce the dimensionless variables $t = \frac{\sigma_1}{\nu}\bar{t}$, $z = H\bar{z}$, $\mathbf{V}_1 = \frac{\chi_1}{M_1 H}\bar{\mathbf{V}}_1$, $\mathbf{V}_2 = \frac{\chi_1}{M_1 H}\bar{\mathbf{V}}_{12}$, $\theta_1 = \gamma_{10} H\bar{\theta}_1$, $\theta_2 = \gamma_{10} H\bar{\theta}_2$, $p_1 = \frac{\rho_0\nu\chi_1}{\sigma_1}\bar{p}_1$, and $p_2 = \frac{\rho_0\nu\chi_1}{\sigma_1}\bar{p}_2$ (the vinculums denote the dimensionless quantities) and determine γ_{10} from the formula

$$\gamma_{10} = \frac{\theta_{10} - \theta_{20}}{h_1 + \dfrac{\lambda_1}{\lambda_2} h_2}\,. \tag{15}$$

In this case, the system of equations (1)–(2) takes the form (the vinculums are removed)

$$\frac{\partial \mathbf{V}_1}{\partial t} = -\,M_1 \nabla p_1 + R_1 \theta_1 \mathbf{e}_z - \mathbf{V}_1\,, \tag{16}$$

$$\mathrm{Pr}_1\frac{\partial \theta_1}{\partial t} = (\mathbf{V}_1 \mathbf{e}_z) + \Delta\theta_1\,, \tag{17}$$

$$\frac{\partial \mathbf{V}_2}{\partial t} = -M_1 \nabla p_2 - \frac{\sigma_1}{\sigma_2}\mathbf{V}_2 + R_2 \theta_2 \mathbf{e}_z\,, \tag{18}$$

$$\mathrm{Pr}_2 \frac{\partial \theta_2}{\partial t} = (\mathbf{V}_2 \mathbf{e}_z) + q\Delta\theta_2\,. \tag{19}$$

Here, $\mathrm{Pr}_1 = \frac{\nu H^2}{\chi_1 \sigma_1}$ is the Prandtl number for a homogeneous snow layer of thickness H; $\mathrm{Pr}_2 = \frac{M_1}{M_2}\frac{\nu H^2}{\chi_1 \sigma} = \frac{M_1}{M_2}\mathrm{Pr}_1$, $q = \frac{\chi_1}{\chi_2}\frac{M_1}{M_2} = \frac{\lambda_1}{\lambda_2}$, $R_1 = \frac{\beta g f_1 H^2 M_1 \sigma_1 \gamma_{10}}{\nu \chi_1}$, and $R_2 = \frac{\beta g f_2 H^2 M_1 \sigma_1 \gamma_{10}}{\nu \chi_1}$ are dimensionless quantities that are analogous to the Rayleigh number Ra.

Equations (5) and (6) in the dimensionless form remain unchanged:

$$\mathrm{div}\,\mathbf{V}_1 = 0\,,\ \ \mathrm{div}\,\mathbf{V}_2 = 0\,. \tag{20}$$

Now, in the system of equations (16)–(20), $\mathbf{V}_1$, $\mathbf{V}_2$, p_1, p_2, θ_1, and θ_2 are dimensionless perturbations, and all the derivatives are taken with respect to the dimensionless coordinates and time. Thus, small perturbations of the air equilibrium are defined by a system of linear homogeneous equations with constant coefficients.

In the system of equations (16)–(19), the pressures p_1 and p_2 as well as the horizontal velocity components V_{1x}, V_{2x}, V_{1y}, and V_{2y} can be eliminated. For this purpose, the operation rot rot will be applied to Eqs. (16) and (17) and the vector equations obtained as a result of this operation will be projected to the z axis. As a result, we will obtain a system of four equations for the vertical velocity components V_{1z} and V_{2z} and the temperature perturbations θ_1 and θ_2:

$$\begin{aligned}
&\frac{\partial}{\partial t}(\Delta V_{1z}) = R_1 \Delta_1 \theta_1 - \Delta V_{1z}\,,\\
&\frac{\partial}{\partial t}(\Delta V_{2z}) = R_2 \Delta \theta_2 - \frac{\sigma_1}{\sigma_2}\Delta V_{2z}\,,\\
&\mathrm{Pr}_1 \frac{\partial \theta_1}{\partial t} = \Delta\theta_1 + V_{1z}\,,\\
&\mathrm{Pr}_2 \frac{\partial \theta_2}{\partial t} = q\Delta\theta_2 + V_{2z}\,,
\end{aligned} \tag{21}$$

where $\Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a plane Laplacian.

The boundary conditions follow from (7)–(14).

In the simplest case where the lower and upper boundaries of a snow cover are impenetrable, the boundary conditions for V_z take the form

$$\text{at}\ \ z = 0\ \ \text{and}\ \ z = 1\quad V_{1z}(0) = 0\ \ \text{and}\ \ V_{2z}(1) = 0\,; \tag{22}$$

if the upper boundary is opened,

$$\text{at}\ \ z = 0\ \ \text{and}\ \ z = 1\quad V_{1z}(0) = 0\ \ \text{and}\ \ V_{2z}'(1) = 0\,. \tag{23}$$

In both cases, we obtain the following conditions for the temperatures:

$$\theta_1(0) = 0\,, \quad \theta_2(1) = 0\,. \tag{24}$$

The conjugation conditions (13)–(15) lead to the relations

$$\begin{gathered}\theta_1\left(\frac{h_1}{H}\right) = \theta_2\left(\frac{h_1}{H}\right),\\ \lambda_1\theta_1'\left(\frac{h_1}{H}\right) = \lambda_2\theta_2'\left(\frac{h_1}{H}\right),\\ f_1\sigma_1 V_{1z}\left(\frac{h_1}{H}\right) = f_2\sigma_2 V_{2z}\left(\frac{h_1}{H}\right).\end{gathered} \tag{25}$$

We will find particular solutions of the system of Eqs. (21), defining the normal perturbations that change exponentially with time and are periodic in the plane (x, y):

$$\begin{gathered}V_{1z}(x, y, z, t) = V_1(z)\exp\left[-\lambda t + i(k_1 x + k_2 y)\right],\\ V_{2z}(x, y, z, t) = V_2(z)\exp\left[-\lambda t + i(k_1 x + k_2 y)\right],\\ \theta_1(x, y, z, t) = \Theta_1(z)\exp\left[-\lambda t + i(k_1 x + k_2 y)\right],\\ \theta_2(x, y, z, t) = \Theta_2(z)\exp\left[-\lambda t + i(k_1 x + k_2 y)\right].\end{gathered} \tag{26}$$

Here, k_1 and k_2 are real wave numbers characterizing the periodicity of perturbations along the x and y directions and $V_1(z)$, $V_2(z)$, $\Theta_1(z)$, and $\Theta_2(z)$ are amplitudes of these perturbations.

Substituting (25) into (21) gives a system of ordinary, linear, homogeneous differential equations for the amplitudes of the perturbations:

$$-\lambda(V_1'' - k^2 V_1) = (V_1'' - k^2 V_1) + k^2 R_1\Theta_1\,, \quad 0 < z < \frac{h_1}{H}\,; \tag{27}$$

$$-\lambda(V_2'' - k^2 V_2) = \frac{\sigma_1}{\sigma_2}(V_2'' - k^2 V_2) + k^2 R_2\Theta_2\,, \quad \frac{h_1}{H} < z < 1\,; \tag{28}$$

$$-\lambda\,\mathrm{Pr}_1\Theta_1 = (\Theta_1'' - k^2\Theta_1) + V_1\,, \quad 0 < z < \frac{h_1}{H}\,; \tag{29}$$

$$-\lambda\,\mathrm{Pr}_2\Theta_2 = (\Theta_2'' - k^2\Theta_2) + V_2\,, \quad \frac{h_1}{H} < z < 1\,. \tag{30}$$

Here, the prime denotes the differentiation with respect to z, and $k^2 = k_1^2 + k_2^2$.

The boundary conditions follow from (22)–(24) and the conjugation conditions (25): when $z = 0$ and $z = 1$ at an impenetrable boundary,

$$V_1(0) = 0\,, \quad V_2(1) = 0\,; \tag{31}$$

if the lower boundary of the snow cover is impenetrable and its upper boundary is penetrable,

$$V_1(0) = 0\,, \quad V_2'(1) = 0\,. \tag{32}$$

The amplitude values of the temperature should satisfy the conditions

$$\Theta_1(0) = 0\,,\quad \Theta_2(1) = 0\,. \tag{33}$$

The conjugation conditions take the form

$$V_2\left(\frac{h_1}{H}\right) = \frac{f_1\sigma_1}{f_2\sigma_2} V_1\left(\frac{h_1}{H}\right), \tag{34}$$

$$\Theta_2\left(\frac{h_1}{H}\right) = \Theta_1\left(\frac{h_1}{H}\right), \quad \Theta_2'\left(\frac{h_1}{H}\right) = \frac{\lambda_1}{\lambda_2}\Theta_1'\left(\frac{h_1}{H}\right). \tag{35}$$

A nontrivial solution of problem (27)–(35) can be obtained only at certain values of λ representing eigenvalues of this problem; the corresponding eigenfunctions are the amplitudes of the perturbations $V_1(z)$, $V_2(z)$, $\Theta_1(z)$, and $\Theta_2(z)$. Thus, the boundary-value problem (27)–(35) determines the spectrum of characteristic perturbations of the air equilibrium in a snow cover.

NOTATION

c_p, heat capacity of air at a constant pressure, J/(kg·deg); c_{s1}, c_{s2}, heat capacities of the snow layers, J/(kg·deg); $\mathbf{e}_z$, unit vector along the z axis; f_1 and f_2, porosity coefficient of the lower and upper snow layers; g, free fall acceleration, m/sec^2; H, total thickness of a snow cover, m; h_1 and h_2, thicknesses of the lower and upper snow layers, m; k, wave number, m^{-1}; k_1 and k_2, real wave numbers along the x and y axes, m^{-1}; L_s, specific heat of ice evaporation, J/kg; p_1 and p_2, pressure of the air contained in the lower and upper layers, Pa; $\bar{p}_1$ and $\bar{p}_2$, dimensionless pressures; $\Pr_1$, $\Pr_2$, Prandtl numbers; R_w, gas constant of the water vapor, J/(kg·deg); T_0 = 273 K; t, time, sec; $\bar{t}$, dimensionless time; $\mathbf{V}_1$ and $\mathbf{V}_2$, mass velocities of the air in the lower and upper snow layers, m/sec; $\bar{\mathbf{V}}_1$ and $\bar{\mathbf{V}}_2$, dimensionless velocities; $V_1(z)$, $V_2(z)$, dimensionless amplitudes of the air-velocity perturbations; V_x, V_y, V_z, velocity components directed along the coordinate axes, m/sec; z, vertical coordinate, m; $\bar{z}$, dimensionless coordinate; β, coefficient of thermal air expansion, deg^{-1}; γ_{10}, γ_{20}, equilibrium temperature gradients in the snow layers, deg/m; Δ, Laplace operator; θ_1 and θ_2, small variations of the temperature in the lower and upper snow layers, $^{\circ}$C; $\bar{\theta}_1$, dimensionless temperature; θ_{1st}, θ_{2st}, stationary temperature fields of the snow, $^{\circ}$C; θ_{10} and θ_{20}, temperature of the lower and upper bases of the snow cover, $^{\circ}$C; $\Theta_1(z)$ and $\Theta_2(z)$, dimensionless amplitudes of the temperature perturbations in the lower and upper snow layers; χ_1, χ_2, thermal diffusivity of the snow, m^2/sec; λ, decrement, sec^{-1}; λ_1, λ_2, heat-conductivity coefficients of the snow, J/(m·sec·deg); ν, kinematic viscosity of the air, m^2/sec; ρ_0, density of the air, kg/m^3; ρ_{s1}, ρ_{s2}, density of the snow layers, kg/m^3; ρ_{w0}, saturated-vapor density at 0°C, kg/m^3; σ_1, σ_2, penetrability coefficients of the snow, m^2. Subscripts: st, stationary; s, snow; w, water vapor.

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